Claw-free circular-perfect graphs

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Abstract

The circular chromatic number of a graph is a well-studied refinement of the chromatic number. Circular-perfect graphs is a superclass of perfect graphs defined by means of this more general coloring concept. This paper studies claw-free circular-perfect graphs. A consequence of the strong perfect graph theorem is that minimal circular-imperfect graphs G have $\min\{\alpha(G), \omega(G)\} = 2$. In contrast to this result, it is shown in [9] that minimal circular-imperfect graphs G can have arbitrarily large independence number and arbitrarily large clique number. We prove that claw-free minimal circular-imperfect graphs G have $\min\{\alpha(G), \omega(G)\} \leq 3$.

Keywords: circular-perfect graph, claw-free graph

Let $G = (V, E)$ be a graph with vertex set V and edge set E, then a *k-coloring* of G is a mapping $f : V \to \{1, \ldots, k\}$ with $f(u) \neq f(v)$ if $uv \in E$, i.e., adjacent vertices receive different colors. The minimum k for which G admits a k-coloring is called the *chromatic number* $\chi(G)$. The *clique number* $\omega(G)$ (resp. *independence number* $\alpha(G)$) of G is the order of a largest clique (resp. independent set) of G, i.e., the maximum number of pairwise adjacent (resp. non-adjacent) vertices of G.

The circular chromatic number and circular clique number of graphs are refinements of the chromatic number and the clique number. Suppose $G = (V, E)$ is a graph with at least one edge, and $k \geq 2d$ are positive integers. A *(k, d)-circular coloring* of G is a mapping $f : V \to \{0, \ldots, k-1\}$ with $d \leq |f(u) - f(v)| \leq k - d$ if $uv \in E$. The *circular chromatic number* $\chi_c(G)$ is the minimum $\frac{k}{d}$ taken over all (k, d)-circular colorings of G. Since every $(k, 1)$-circular coloring is a usual k-coloring of G, we have $\chi_c(G) \leq \chi(G)$. On the other hand, it is known [13] and easy to see that for any graph G, $\chi_c(G) > \chi(G) - 1$, and hence $\chi(G) = \lceil \chi_c(G) \rceil$. So $\chi_c(G)$ is a refinement of $\chi(G)$.

Let $K_{k/d}$ with $k \geq 2d$ denote the graph with the k vertices $0, \ldots, k-1$ and edges ij such that $d \leq |i - j| \leq k - d$. The graphs $K_{k/d}$ are called *circular cliques*. Circular cliques include all cliques $K_t = K_{t/1}$, all odd antiholes $\overline{C}_{2t+1} = K_{(2t+1)/2}$, and all odd holes $C_{2t+1} = K_{(2t+1)/t}$. The *circular clique number* is defined as $\omega_c(G) = \max\{\frac{k}{d} : K_{k/d} \subseteq G, \gcd(k, d) = 1\}$. It follows from the definition that $\omega(G) \leq \omega_c(G)$. It is also known [17] that for any graph G, $\omega_c(G) < \omega(G) + 1$, and hence $\omega(G) = \lfloor \omega_c(G) \rfloor$. Therefore $\omega_c(G)$ is a refinement of $\omega(G)$.

Obviously $\omega(G)$ is a lower bound for $\chi(G)$. A graph G is *perfect* if each induced subgraph $G' \subseteq G$ has $\omega(G') = \chi(G')$. Similarly, $\omega_c(G)$ is a lower bound for $\chi_c(G)$. A graph G is called *circular-perfect* [17] if each induced subgraph $G' \subseteq G$ has $\chi_c(G') = \omega_c(G')$.

Perfect graphs have been studied extensively since the concept and two conjectures (the weak and the strong perfect graph conjectures) were proposed by Berge [1] in 1961. The weak perfect graph conjecture was settled by Lovász [8] in 1972. Recently, the strong perfect graph conjecture has been settled by Chudnovsky, Robertson, Seymour and Thomas in [2], which gives a characterization of perfect graphs by means of forbidden induced subgraphs: a graph G is perfect if and only if G contains neither chordless odd cycles C_{2k+1} with $k \geq 2$, nor their complements $\overline{C}_{2k+1}$.

It follows from the definitions that for any graph G, $\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq$

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$\chi(G)$. Therefore every perfect graph is circular-perfect. However, odd cycles and their complements are circular-perfect graphs but not perfect graphs. So the class of circular-perfect graphs is a proper superclass of the class of perfect graphs.

Is there a simple characterization of circular-perfect graphs by means of forbidden induced subgraphs? It is shown in [14] that the line graph $L(G)$ of a cubic graph G is circular-perfect if and only if G is 3-edge colourable. Thus such a characterization of circular-perfect graphs implies a characterization of critically non-3-edge colourable cubic graphs, which is known to be a difficult problem. So it is unlikely that there is a simple forbidden induced subgraph characterization of circular-perfect graphs. Some sufficient conditions for a graph to be circular-perfect were obtained in [16,17]. Classes of (minimal) circular-imperfect graphs were constructed in [9,11,12,15]. Minimal circular-imperfect line graphs were studied in [14]. In this paper, we study claw-free circular-perfect graphs.

A graph G is *claw-free* if $K_{1,3}$ is not an induced subgraph of G. Claw-free graphs is a superclass of line graphs and has been studied extensively in the litterature. Recently, Chudnovsky and Seymour [4,3] presented a structural characterization of claw-free graphs. A graph G for which the neighbourhood of each vertex can be covered by two cliques is called a *quasi-line graph*. We use their characterization, restricted to quasi-line graphs, to prove a structural property of minimal circular-imperfect graphs. One consequence of the strong perfect graph theorem is that minimal imperfect graphs G have $\min\{\omega(G), \alpha(G)\} = 2$. It was asked in [11] whether $\min\{\omega(G), \alpha(G)\}$ is bounded for minimal circular-imperfect graphs G. This question was answered in the negative in [9], where it is proved that for any positive integer k, there is a minimal circular-imperfect graph G with $\min\{\omega(G), \alpha(G)\} \geq k$.

We show that if restricted to claw-free graphs, the question above has a positive answer: if G is a claw-free minimal circular-imperfect graph, then $\min\{\omega(G), \alpha(G)\} \leq 3$.

Before the strong perfect graph conjecture becomes a theorem, the conjecture was confirmed for claw-free graphs in [7][10]. Our result above implies an alternative proof of this result, of course without making use of the strong perfect graph theorem [2].

Sketch of the proof

Suppose G is a claw-free graph with independence number at least 3. It was proved by Fouquet [6] that for any vertex x of G, the neighborhood $N_G(x)$ of x either contains an induced C_5, or can be covered with two cliques. If G is circular-perfect, then $N_G(x)$ does not contain an induced C_5, for otherwise G contains the odd wheel

W_5, which is circular-imperfect. Thus we have the following observation: *if G is a claw-free circular-perfect graph with independence number at least 3, then G is a quasi-line graph.*

It turns out that claw-free circular-perfect graphs with an induced odd antihole of size at least 7 have a basic structure:

Theorem 1. *If G is a connected claw-free circular-perfect graph with an induced odd antihole H of size at least 7 then $G \setminus H$ is a clique. Furthermore $\alpha(G) = 2$.*

Since every minimal circular-imperfect graph is 2-connected, we have the following corollary:

Corollary 1. *If G is a claw-free minimal circular-imperfect graph and contains an induced odd antihole H of size at least 7, then $\alpha(G) \leq 3$.*

It remains to study claw-free circular perfect graphs G that do not contain an odd antihole of order at least 7. Due to Theorem 1, G has independence number at least 3, and is therefore, as mentioned above, quasi-line. We establish that if G has clique number at least 4 then G has an independent set I that intersects each maximum clique.

We prove a stronger statement:

Theorem 2. *If G is a quasi-line graph, $\omega(G) = k \geq 4$ and for every vertex x, $G - x$ has a k-colouring, then either G is the complement of a circular clique or G has a stable set which intersects every maximum clique of G.*

As a consequence, a claw-free graph G with $\omega(G) = k \geq 4$ and $\alpha(G) \geq 4$ can not be minimal circular-imperfect. Because otherwise, G is not the complement of a circular-clique [5], is quasi-line since it does not contain the odd wheel W_5 (which is already minimal circular-imperfect). Hence there is an independent set I intersecting each maximum clique. Since $G - I$ is circular-perfect, we have $\omega_c(G - I) = \chi_c(G - I)$. Due to Corollary 1, G, and thus $G - I$, does not contain $K_{(2p+1)/2}$ for $p \geq 3$. It follows that $\omega_c(G - I) = \omega(G - I) = k - 1$ and hence $\chi(G - I) = \chi_c(G - I) = k - 1$. But then $\chi(G) = \omega(G) = k$, and hence G is circular-perfect.

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